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The Use of Electronic Computers to Solve Subsurface Drainage Problems

George S. Taylor and J. N. Luthin

UNIVERSITY OF CALIFORNIA DIVISION OF AGRICULTURAL SCIENCES

The procedure for using the computer to solve steady-state drainage problems is described in detail. Flow charts are presented for the solution of problems that include stratified soils.

The precision of the results as a function of mesh size is discussed. The cost of the computer program as a function of the desired precision is also discussed.

The usefulness of the computer is illustrated by several solutions for ponded flow into drains in stratified soils. The potential use of the computer in other subsurface drainage problems is discussed and a proposal is made for obtaining solutions of the falling water table case in tile drainage.

THE AUTHORS:

George S. Taylor is Professor of Agronomy, Ohio State University and Ohio Agricultural Experiment Station.

J. N. Luthin is Professor of Irrigation and Irrigationist in the Agricultural Experiment Station, Davis.

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The Use of Electronic Computers to Solve Subsurface Drainage Problems[1]

SUBSURFACE DRAINAGE systems are evaluated by field experimentation, laboratory studies involving tank models, analog systems such as electrical resistance networks, and analytical and numerical analyses. In this report our primary concern is with numerical analysis. In such studies, boundary and internal soil conditions are specified with respect to hydraulic-head potential ϕ and soil hydraulic conductivity K. The applicable flow equation relating ϕ and K in the soil is then solved by an iterative procedure. From the resulting data one obtains usable information concerning water flow into drains.

The application of numerical analysis to certain problems in subsurface drainage was reported about a decade ago by Luthin and Gaskell (1950) and by Kirkham and Gaskell (1950). Because of the extensive calculations required, numerical analysis has been used to only a limited extent since that time. The availability of high-speed computers removes this obstacle and permits extensive use of numerical analysis in all types of soil-moisture flow problems.

The primary objective of this report is to illustrate the use of high-speed computers for studying moisture-flow problems encountered in drainage. The drainage case selected for these analyses is that of ponded flow into drains which are embedded in stratified soil. While the ponded flow case has limited practical application, it is treated here because analytic solutions are available for comparison with the computer results. Also the effect of soil stratification on ponded flow into drains can be explored for some cases which do not lend themselves easily to solution by analytical analysis.

PROCEDURES

The drainage case analyzed in this study is shown in figure 1. Drain tubes of radius r are buried at a depth d in saturated soil and are running full with no back pressure. The drains are essentially horizontal and their walls are infinitely permeable. The drains are considered to be of infinite length so that flow into the drains is of two-dimensional character. The soil is layered but isotropic with respect to its hydraulic conductivity K. The ground surface is covered with a continuously maintained thin film of water. The drain depth d, the interfaces at L_1 and L_2, and the distance to the impermeable layer h are variables. Likewise the hydraulic conductivities K_1, K_2, and K_3 may assume different values.

For steady-state laminar flow of fluid

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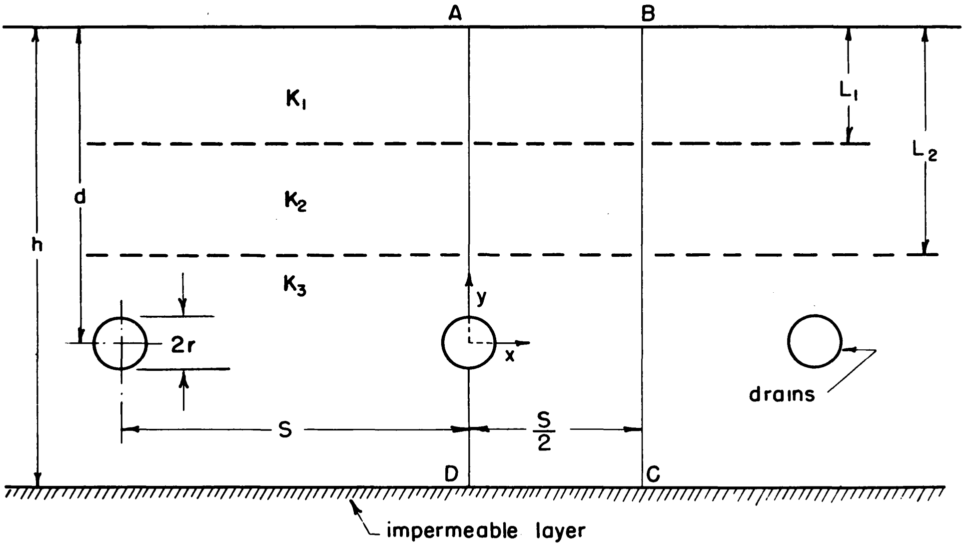

Fig. 1. Schematic representation of the drainage case studied in this paper. Drains of radius r are embedded at a depth d in a 3-layered soil which is underlain by an impermeable layer at a depth h. The hydraulic conductivities of the top, middle, and bottom layers are K_1, K_2 and K_3, respectively. A water table is maintained at the soil surface and the drains are running full with no back pressure.

in a saturated porous medium, the appropriate flow equation for two-dimensional analysis is given by equation [1].

$$\frac{\partial}{\partial x}\left(K\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial \phi}{\partial y}\right) = 0 \qquad [1]$$

In this expression K is the hydraulic conductivity of the porous medium and ϕ is the hydraulic head. The hydraulic conductivity K is defined by equation [2], where v is the macroscopic flow velocity per unit cross-sectional area, and gradient ϕ is evaluated in the direction of greatest change in ϕ

$$v = -K \text{ grad. } \phi \qquad [2]$$

For the flow problem shown in figure 1, $K = K(x, y)$ must be used at points along an interface between layers of different hydraulic conductivity. At other points in the flow region, K is not a space function and equation [1] reduces to the familiar Laplace equation in two dimensions as shown by equation [3].

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad [3]$$

As used in this study the potential ϕ is that given by equation [4], where y is the elevation above the drain center of a point $P(x, y)$, and H is the gauge or hydrostatic pressure expressed in terms of a water-column height.

$$\phi = y + H \qquad [4]$$

Of particular interest in this study are the potentials at the soil surface and at the drain. By definition the gauge pressure H is zero along the water table, and ϕ is numerically equal to the drain depth d at the ground surface. Since the drains are running full with no back pressure, the sum of y and H along the drain circumference is always equal to zero since the gauge pressure is zero at the center of the drain, $+ r$ at the bottom, and $-r$ at the top.

Earlier reports have shown methods by which numerical analysis can be ap-

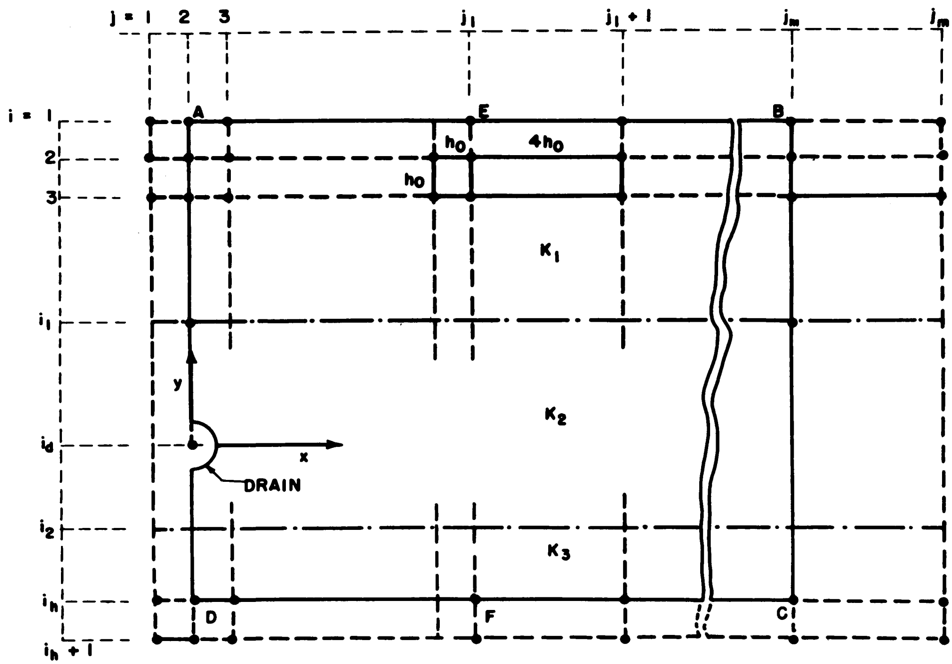

Fig. 2. Representation of the network of rectangular meshes superimposed over the area *ABCD* shown in figure 1. Square meshes of dimension h_0 are used between the planes *AD* and *EF*, while meshes of dimension h_0 by $4h_0$ are used in the remaining area. The vertical plane *EF* may occur at $j_1 = 5, 6, \cdots$, or $j_m - 2$. The columns $j = 1$ and $j = j_m + 1$ and the row $i = i_h + 1$ serve as reflexive boundaries to fulfill the condition that *AD*, *BC* and *CD*, respectively, are streamlines.

plied to steady-state flow problems such as the one illustrated in figure 1. The procedure utilized herein is essentially that reported by Luthin and Gaskell (1950) and by Kirkham and Gaskell (1950). The major difference is that numerical calculations are done by a high-speed electronic computer rather than with a desk calculator. The problem is solved by first drawing a rectangular grid over the region *ABCD* as shown in figure 2. Only half of the region on one side of the drain is needed because of symmetry. The left-hand portion of the region *ABCD* is represented by square meshes and the remaining portion by rectangular meshes which have a horizontal dimension four times that of the vertical. This particular arrangement is followed as a time-saving and economy feature, since previous studies have shown that potential changes in the horizontal direction are quite small except near the drain. For computer programming, each node (or grid point) is identified by the subscripts $_{i,\ j}$.

The known value of $\phi = d$ is assigned to nodes along the soil surface. The fixed potential at the drain circumference (that is, $\phi = r$) is assigned to the node at the drain center. All other nodes are initially assigned a value of $\phi = 0$. These latter values are then repeatedly altered by an iterative procedure until equation [1] is approximately satisfied on a finite (but small) scale. This is done by traversing the nodes repeatedly and systematically, the value at each node (other than ones of fixed value) being replaced by a calculated one. The calculated value is based on the magnitude of those at its four neighboring nodes, the distances to the four nodes, and the hydraulic conductivity in the intervening intervals. (Special formulas for making these calculations are given in the following section.) When the iterative process is continued until successive traverses bring about only small changes in ϕ, the calculations are discontinued and the problem is considered to be "solved." The resulting values of ϕ can then be used to determine such quantities as equipotentials, streamlines, flow velocities, and drain flow rates. From these latter quantities, the effectiveness of drainage-system design for various hydraulic-conductivity depth profiles can be evaluated.

Formulas for Rectangular Meshes in Stratified Soil

For the grid shown in figure 2 there are four cases for which approximation formulas must be derived in order to solve equation [1] by numerical analysis. These cases are illustrated in figure 3. The formulas are given by equations [5], [6], [7], and [8], respectively, for the cases shown in figure 3*a*, *b*, *c*, and *d*. These equations are given in terms of the center (zero) node. They apply for a node located either in the interior of a homogeneous layer or on an interface which separates an upper layer of hydraulic conductivity K_U from a lower one of conductivity K_L. For nodes in the row $i = i_1$ (see figure 2), R is designated as R_1 and is equal to K_1/K_2; while in row i_2, R is given by K_2/K_3. In equation [6] the parameter C is a function of the ratio h_0/r, where h_0 is the dimension of the square mesh at the drain and r is the drain radius. A table of C values is given in the Appendix for some values of h_0/r which are commonly encountered in the numerical analysis of drainage problems. If the node is not located on an interface, R is unity and then equations [5] through [8] become finite difference approximations of equation [3].

The finite difference equations for the three cases shown in figure 3*a*, *c*, and *d* are based on a linear rate of change in the potential ϕ between the node at 0 and 1, 0 and 2, and so forth. These equations are taken directly from Vimoke and Taylor (1962; Eq. [30]). In using the formula of Vimoke and Taylor to obtain equation [7], we set $b = d = a$, c/a equal to M, m equal to R, and V equal to ϕ. By making M equal to unity, equation [5] then results from equation [7]. In using the formula of Vimoke and Taylor to obtain equation [8], we set $b = d$, a/d and c/d equal to M, m equal to R, and V equal to ϕ. When R and M are both unity, [5], [7], and [8] yield the

Case 1. Node on interface, square mesh (see figure 3*a*):

$$\phi_0 = [2R\phi_1 + (R + 1)(\phi_2 + \phi_4) + 2\phi_3]/4(R + 1) \quad [5]$$
$$\text{where } R = K_U/K_L$$

Case 2. Same as Case 1, except adjacent to drain (see figure 3*b*):

$$\phi_0 = [2R\phi_1 + (R + 1)(\phi_2 + \phi_4/C) + 2\phi_3]/(3 + 1/C)(R + 1) \quad [6]$$
$$\text{where } C = f(h_0/r) \qquad \text{(See Appendix table A-1)}$$

Case 3. Node on interface, change in mesh size (see figure 3*c*):

$$\phi_0 = [M(1 + M)(R\phi_1 + \phi_3) + (1 + R)(\phi_2 + M\phi_4)]/(1 + M)^2(1 + R) \quad [7]$$

Case 4. Node on interface, rectangular mesh (see figure 3*d*):

$$\phi_0 = [2RM^2\phi_1 + (1 + R)(\phi_2 + \phi_4) + 2M^2\phi_3]/2(1 + R)(1 + M^2) \quad [8]$$

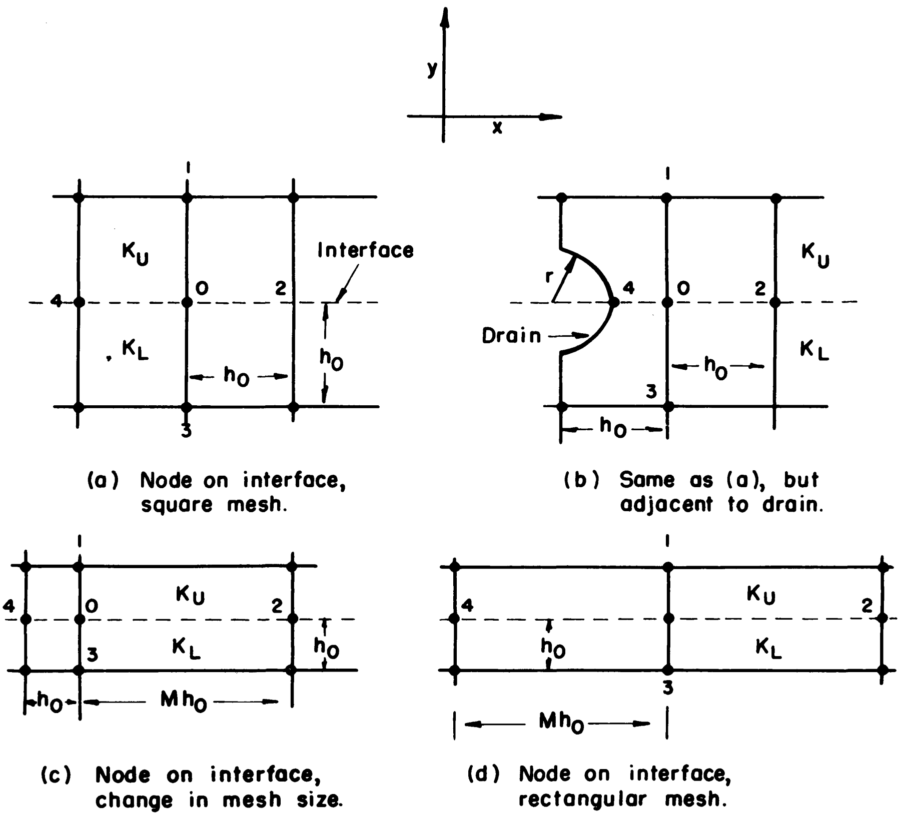

Fig. 3. Sketches of the four cases for which formulas are needed to solve equations [1] and [2] by numerical analysis. In all cases a horizontal interface separates an upper layer of hydraulic conductivity K_U from a lower one of conductivity K_L.

same results, namely the appropriate equation for a square mesh in a homogeneous soil.

Equation [6] is based on a logarithmic change in ϕ between 0 and 4 and a linear change in ϕ between 0 and the remaining three nodes. This equation is obtained directly from Vimoke and Taylor (1962; Eq. [53]). In using the formula of Vimoke and Taylor to obtain equation [6], we set m equal to R, V equal to ϕ, and C_d equal to C.

Equations [5], [7], and [8] are essentially the same as those utilized by other investigators who assume a linear rate of change in ϕ between adjacent nodes. Equation [5] is the same expression as one derived for this case by Luthin and Gaskell (1950; Eq. [6]). When R is unity, equation [5] is identical to Luthin and Gaskell's equation [2], both being for a square mesh in a homogeneous and isotropic soil. When R is equal to unity, equation [8] is the same expression as one given by Kirkham and Gaskell (1950; Eq. [8]) for the case shown in figure 3*d*. Equation [6] is similar in form to one given by Luthin and Gaskell (1950; Eq. [4]) for the case shown in figure 3*b* and for R equal to unity. However, equation [6] differs significantly from Luthin and Gaskell's and from one used by Isherwood (1959), in that equation [6] is based on a logarithmic change in potential between the nodes 0 and 4. The other two investigators assume a linear relationship. Vimoke, *et al.* (1962), show that a logarithmic rather than a linear interpolation is necessary to bring satisfactory agreement between results of numerical analyses (or network analogs) and exact mathematical solutions. The reader is referred to the work of Vimoke, *et al.*, for a detailed discussion of the case shown in figure 3b and an evaluation of the parameter C.

Computer Programming

Before discussing the computer program used to solve the problem shown in figure 2 by numerical analysis, one might first consider the manner in which calculations are made. As indicated previously, known values of ϕ are assigned to boundary nodes and a ϕ value of zero is initially assigned to all others. For each node other than boundary ones, a new potential ϕ is calculated by one of the following equations, [5], [6], [7], or [8]. The particular equation utilized depends on the location of the nodes as shown in figure 3. The calculations are made first for the column $j = 2$, starting with $i = 2$ and proceeding through $i = i_h$. This process is then repeated for columns $j = 3, 4, \cdots, j_m$.

To identify and store the values of ϕ in the computer, it is convenient to designate ϕ as ϕ^n (i, j). (The superscript does not appear in the computer program.) The latter term represents the value of ϕ_0 after the n^{th} iteration for a node whose location is specified by the subscripts $_i$, $_j$. Following the calculation of each ϕ^n (i, j), a quantity Y' is calculated by utilizing equation [9].

$$Y' = W(\phi^n(i,j) - \phi_{(i,\,j)}{}^{n-1}) = WY \qquad [9]$$

In this expression, Y is the residual term. It represents the difference between the most recent value of ϕ (that is, the n^{th} iteration) and the one which was obtained in the previous $(n - 1)$ iteration. The parameter W is an over-relaxation constant, reported by Young (1954; 1956) and by Young and Lerch (1953), which reduces the number of iterations for a prescribed level of precision. Its magnitude is determined by the number of nodes and ranges between 1.0 and 2.0. The quantity Y' is referred to as "RESIDUAL" by us but differs from Y by the factor W. The magnitude of Y indicates the rapidity with which the potential at the node i, j is changing from the $(n - 1)$ to the n^{th} iteration. Small values of Y indicate, for example, that subsequent iterations will bring about only small improvements in precision. During the n^{th} iteration, the absolute value of Y' is summed for all the i, j's. If the summed values of Y' are less than a prescribed value, say DELTA, the iterative process is discontinued. If not, the $(n + 1)$, $(n + 2)$, and so forth, iterations are performed until the summed values of Y' are reduced to the magnitude of DELTA.

A flow chart is shown in figure 4 for programming the problem illustrated in figure 2. In analyzing the chart, the reader is reminded that this particular chart is not unique for the problem at hand. Some modification may even be necessary if one uses a different computer. The flow chart is given here so that the interested reader may comprehend the overall scheme by which such problems are handled.

Prior to the first step shown in the flow chart, the following parameters must be specified and "read" into the computer: i_1, i_2, i_d, i_h, j_l, j_m, R_1, R_2, C, DELTA, and D. The values of i and j specify the physical dimensions of the problem as shown in figure 2, while the values for R and C apply to equations [5] through [8]. The parameter DELTA specifies the precision to which the iteration process is to be carried. The quantity D is the potential assigned to the

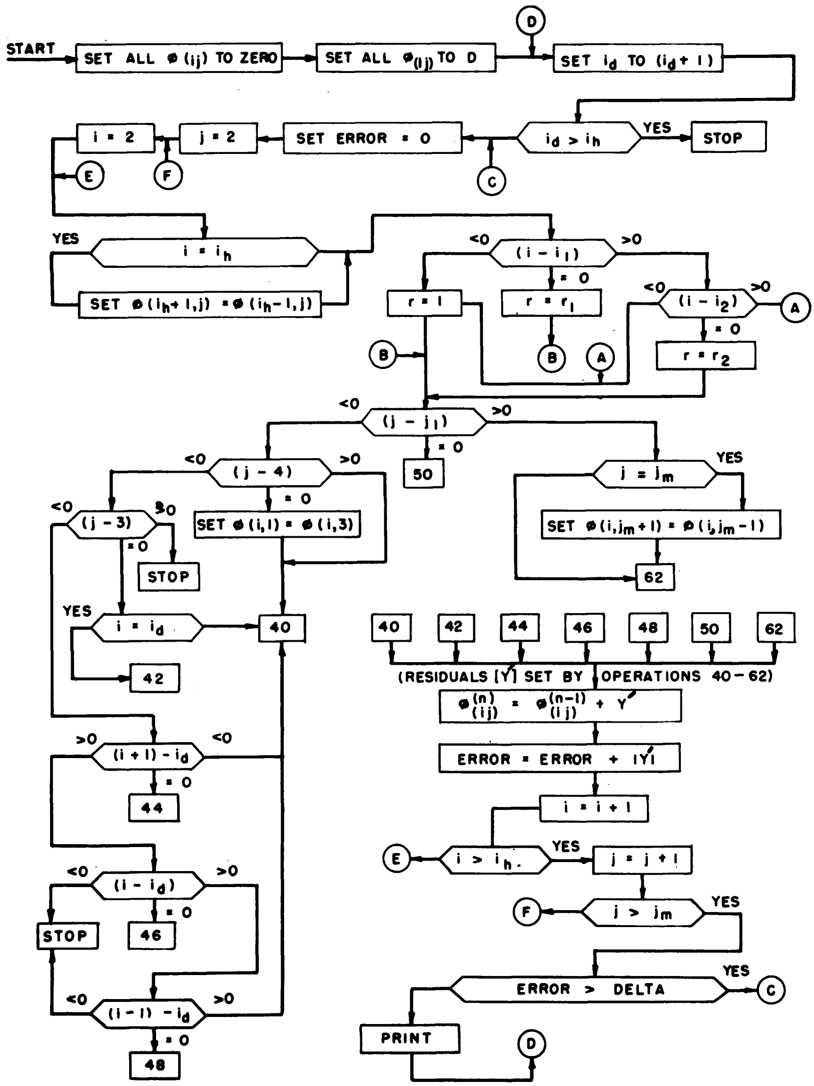

Fig. 4. Flow chart for programming the numerical analysis problem illustrated in figure 2 on an IBM 704 electronic computer.

ground surface, where $D = A(d - r)$ and A is a constant.

In the first step given in figure 4, ϕ is initially set to zero in each case. Secondly, ϕ along the soil surface is then changed to D in each case. (The magnitude of D should not be confused with the encircled letter D which denotes a "step" in the program.) The drain depth is then increased a distance equal to h_0 (see figure 2) by replacing i_d with $(i_d + 1)$. For example, if one wishes to have the first drain depth[2] at the node, for example, $i = 3$, one sets the initial value of $i_d = 2$. (Ignore the encircled letters A through D for the moment.)

[2] A more flexible arrangement which has been used in subsequent studies is to replace i_d by by $(i_d + i_n)$, where i_n is an integer which denotes the interval between drain locations and which must also be initially "read" into the computer.

In the fourth step, i_d is compared with i_h. If i_d is larger than i_h, the program is stopped. Otherwise, the summed values of Y' (that is, ERROR) are set to zero and the iteration begun. This is done by initially setting $j = 2$ and $i = 2$. As will become clear later on in the program, steps E and F increase i and j in steps by units of one, respectively, until all i's and j's are utilized in the iteration.

The current value of i is then compared to i_h. If equal, the potential at the node $(i_h + 1, j)$ is replaced by that at $(i_h - 1, j)$. This operation fulfills the condition of zero flux across the plane DC since the potentials directly above and below this plane are equal. At the next step, the value of i is compared first with i_1 and then with i_2. These comparisons result in the appropriate value of R being assigned to equations [5] through [8]. The value of j is then compared to j_1. If equal to j_1, a RESIDUAL Y' is calculated (operation number 50) by using equation [9]. In this particular case, $\phi^n(i, j)$ is given by equation [7], which is the appropriate equation for nodes in the column $j = j_1$. The value of ϕ is then obtained from the sum of $(\phi_{(i,j)}{}^{n-1} + Y')$ and then stored in the computer. The magnitude of ERROR is computed by adding the absolute value of Y' to the previous value of ERROR. At this step, the value of i is increased by 1, and so forth; however, let us return for the moment to the comparison of j with j_1.

If j is greater than j_1, a comparison is first made with j_m. If $j = j_m$, the potential at the node $(i, j_m + 1)$ is replaced by the potential at $(i, j_m - 1)$. This operation fulfills the condition of zero flux across the plane BC. Operation 62 then directs the calculation of Y', in which $\phi_{(i,j)}{}^n$ is given by equation [8]. If j is less than j_1, then j is compared with $j = 4$. If j is equal to or greater than 4, operation 40 directs the calculation of Y by using equation [5] to yield $\phi_{(i,j)}{}^n$. If $j = 4$, the potential at $(i, 1)$ is first replaced by that at $(1, 3)$ before proceeding with operation 40. This procedure fulfills the condition for zero flux across the plane AD.

For j less than 4, a comparison is first made between it and $j = 3$. If $j = 3$, a second comparison is then made between i and i_d. For $i = i_d$, operation 42 directs the use of equation [6] in calculating Y'. For other values of i, operation 40 is carried out. If j is less than 3 (that is, $j = 2$), then three additional steps are included to determine the location of the node $(i, 2)$ with respect to the node $(i_d, 2)$. If $i = i_d$, the magnitude of Y' is arbitrarily set to zero by operation 46. Since the potential at the drain was initially set to zero, this operation insures that the potential always remains zero. Operations 44 and 48 utilize a modified form of equation [6] in calculating Y' for the case where $(i, 2)$ is either directly above or below $(i_d,2)$.

After ERROR has been accumulated for the current value of i, then i is replaced by $(i + 1)$. If the new value of i is less than or equal to i_h, step E directs the iteration to continue. When i exceeds i_h, j is replaced by $(j + 1)$. If j does not exceed j_m, the iteration then continues for all i's and subsequently on to the next j. When j exceeds j_m, a comparison is then made between ERROR and DELTA. If ERROR exceeds DELTA, step C directs that ERROR again be set to zero and the entire iteration for i, j be repeated in each case. If ERROR does not exceed DELTA, the value of all the potentials is printed, i_d is set to $(i_d + 1)$ and all steps repeated until i_d is again increased. When i_d exceeds i_h, the entire program is stopped.

RESULTS AND DISCUSSION

Mesh Size and Residuals

To evaluate the effect of mesh size and residual (Y) on the computer results, drain flow rates Q are determined for the following drainage situation and compared with those calculated by the exact solution of Kirkham (1949; Eq. [11]). Water flows under saturated conditions through a homogeneous and isotropic soil and into a buried drain (see figure 1). The drain diameter, depth, and spacing are 6 inches, 15 feet, and 48 feet, respectively. An impervious layer is at 21 feet, and the water table is maintained at the soil surface. The drain is running full without back pressure. In the computer analysis, a square grid is used on the flow region[3], and different mesh sizes are employed. The limiting magnitude of the summed RESIDUALS Y' (that is, DELTA) is also varied for each analysis. For each case studied, the drain flow rate Q is determined by summing the increments of flux which enter the soil surface. This evaluation is made by first dividing the horizontal distance between $x = 0$ and $x = S/2$ (see figure 1) into N equal increments of width Δx. The expression in equation [10] is then utilized to obtain Q.

$$Q = 2K \sum_{m=1}^{N} (d\phi_m/dy)\Delta x_m \cdots \quad [10]$$

In this expression K is the conductivity at the ground surface (equal to K_1 in layered soils), and $d\phi_m/dy$ is the potential gradient at the soil surface and in the interval Δx_m.

The effect of mesh size and average residual $\bar{Y}$ on the ratio Q/K is shown in figure 5. For purposes of discussion it is assumed that K is equal to unity, and Q is thus numerically equal to the ratio Q/K. The "average residual" $\bar{Y}$ is obtained by summing the absolute values of all Y's and dividing by the number of nodes. Although the residual Y varies from one node to another, an average value of Y is sufficient for the present discussion. As can be seen from figure 5, a linear relationship exists between Q and $\bar{Y}$. Q was not determined for an extremely large range of $\bar{Y}$; however, a linear relationship is found over a wider range of $\bar{Y}$-values than shown in the graph. As the mesh size is reduced, Q is greater for a comparable value of $\bar{Y}$. There is a relatively large increase when the mesh size is reduced from 3.0 to 1.5 feet, while a small increase is obtained when the mesh size is further reduced to 1.0 foot.

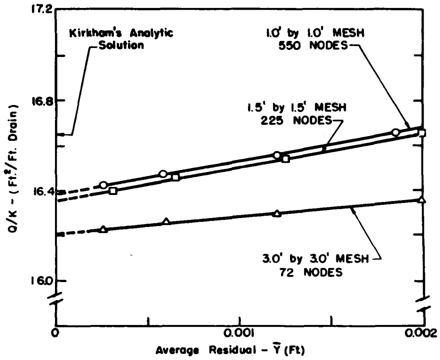

Fig. 5. The effect of residuals and mesh size on the ratio Q/K for ponded flow in a homogeneous and isotropic soil. The drain diameter, depths, and spacing are 6 inches, 15 feet, and 48 feet, respectively. An impervious layer is at 21 feet. The quantity Q/K is evaluated by equation [10] from data obtained with an IBM 704 computer.

The data resulting from numerical analyses such as used here are obtained with greater precision when the resid-

[3] These particular analyses are made with a slightly modified program of that illustrated in figures 2 and 4. In the modified program, j_1 is made larger than j_m so that a square mesh is used in the entire region.

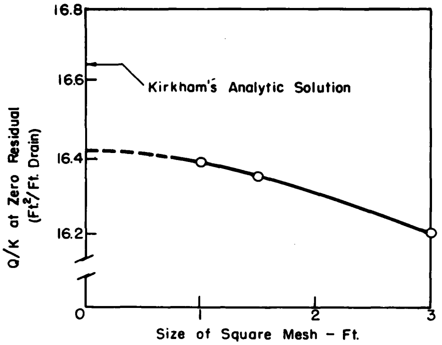

Fig. 6. The effect of mesh size on the ratio Q/K when the latter is evaluated at zero residual. The quantity Q/K is the intercept value of the curves in figure 5.

uals are small. Thus for a particular mesh size, Q can be obtained with high precision when the curves in figure 5 are extrapolated to zero residuals. Accordingly, Q-values of 16.2 and 16.4 are obtained, respectively, for square mesh sizes of 3.0 and 1.0 feet.

If one obtains Q as described above and plots these values against mesh size (see figure 6), the accuracy of Q can be evaluated by first extrapolating the curve to zero mesh size and then comparing the intercept value to that obtained by an exact analytical solution.

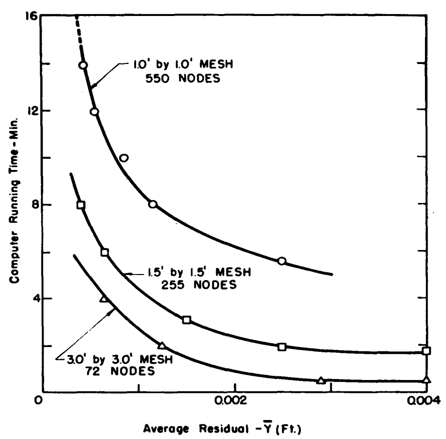

Fig. 7. The effect of mesh size and the average residual $\bar{Y}$ on computer running time. The drainage case evaluated is identical to that used to obtain the results shown in figure 5.

This is done in figure 6 by utilizing the data presented in figure 5. A nearly straight line is obtained. The intercept value is 16.42 as compared with Kirkham's analytic value of 16.65, and the deviation from the latter is -1.5%.

The error in the potential is greatest at the nodes near the drains. An unknown error is contributed by "rounding off." Eight significant figures are used in the computer calculations.

Residuals, Mesh Size, and Computer Running Time

The effect of residuals and mesh size on computer running time is illustrated in figure 7. As used here, "running time" is the time required to yield the potentials at all nodes after the boundary conditions are specified. It includes the time to read punched cards, carry out the required computations, and print the results. As one might predict, a decrease in mesh size or a reduction in the residual Y increases the running time. The running time increases quite rapidly as the average residual approaches zero. The horizontal portion of the curves results because a high percentage of the computer running time is taken in reading and printing, this being a "fixed" quantity as compared to computation time. If the rental charge of available computers is known, the information given in figure 7 for an IBM 704 can be used to estimate the cost of utilizing a computer in numerical analysis. Running time is primarily dependent on the number of nodes and the level of precision sought. For a given level of precision, running time is roughly proportional to n^3, the number of nodes. The number of nodes to be used depends also on the rapid-access-memory size of the available computer.

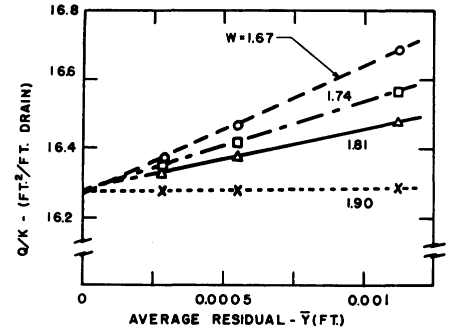

Fig. 8. The effect of the overrelaxation constant W on the ratio Q/K for different values of the average residual $\bar{Y}$. The drainage case evaluated is that reported in figure 5. A square mesh of dimensions 1.0 by 1.0 feet is utilized in the numerical analysis program, resulting in 21 nodes in the vertical direction and 25 in the horizontal.

Overrelaxation Constant

A considerable reduction in computer running time can be obtained by the use of an overrelaxation constant (see equation [9]). The effect of different values for the constant W on the rate of convergence of Q/K and on running time is shown in figure 8 and table 1. All the curves in figure 8 are straight lines and yield similar values for Q/K when the curves are extrapolated to zero residual. The running time is about four times greater, however, for the lowest value of W than for the highest.

Young (1956) has reported optimum values of W for the case where a square mesh is used and the region of interest is also square. Some of these values are tabulated in the Appendix, table A-2. If one does not wish to use refined techniques for deciding on the optimum value of W, an evaluation of the type illustrated in figure 8 and table 1 will often suffice. In such an evaluation the value of W will be decided on the basis of computer running time and rapidity of convergence of potential ϕ. In figure 8, for example, rapid convergence is indicated by the nearly horizontal curve for $W = 1.90$.

TABLE 1

EFFECT OF THE OVERRELAXATION CONSTANT W ON COMPUTER RUNNING TIME*

Overrelaxation constant W	Average residual $\bar{Y}$	Computer running time
	feet	*minutes*
1.36	.....	....
1.67	.00563	8.0
1.81	.00563	4.5
1.90	.00563	3.0
1.36	.00113	12.0
1.67	.00113	7.0
1.81	.00113	3.5
1.90	.00113	3.0

* The drainage case evaluated is that which yields the information reported in figure 5.

Expanding the Mesh Size

The effect of expanding the mesh size in ponded flow analyses is illustrated in table 2. As shown therein, only small deviations are introduced by expanding the mesh. In addition, the potentials are not significantly altered by expanding the mesh (data not shown). The use of an expanded mesh reduces the number of nodes and thus the running time for a prescribed residual Y. Thus a considerable savings in computer running time can be realized by utilizing an ex-

TABLE 2

EFFECT OF EXPANDING FROM A SQUARE MESH OF 1.0 BY 1.0 FEET TO A RECTANGULAR MESH OF 4.0 FEET (HORIZONTAL) BY 1.0 FOOT ON THE RATIO Q/K*

Mesh expanded at	Q/K†	Deviation‡
feet	*square feet/ foot of drain*	*per cent*
4	16.36	0.5
8	16.32	0.2
12	16.30	0.1
16	16.28	0
20	16.28	0
24 (midplane)	16.28	0

* The drainage case evaluated is that used to obtain the results shown in figure 5. The mesh is expanded at the vertical plane EF as shown in figure 2.

† Obtained by extrapolating to zero residual as illustrated in figure 5.

‡ Compared to Q/K obtained by a computer analysis using a square mesh of 1.0 by 1.0 feet.

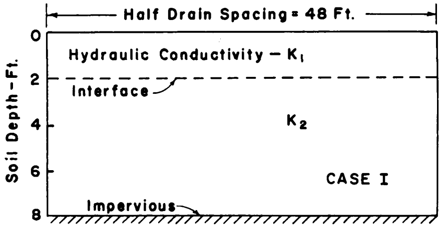

Fig. 9. Drainage situation analyzed by computer.

panded mesh in much of the flow region (see figure 7). No attempt has been made in this study to evaluate the case for a rectangular mesh in the entire region of flow. An approximation formula for a drain in a square mesh has been tested (Vimoke, *et al.*, 1962) and found to be of suitable accuracy. (See figure 3*b* and equation [6].) A similar formula must also be evaluated for a drain in a rectangular mesh before such analyses can be made.

Solutions Obtained for a Stratified Soil

The drainage situation studied is shown in figure 9. The soil represented here consists of two layers, a shallow top layer overlying one of much greater thickness. The hydraulic conductivities in the two layers are assigned different values; however, the conductivity of the top layer either exceeds or is equal to that in the bottom one. Each layer is isotropic with respect to its conductivity. The particular dimensions used are somewhat arbitrary: A depth of 2 feet to the layer interface was chosen so that the boundary conditions correspond roughly to shallow soils which are underlain by a less permeable layer. The depth to the impervious barrier was selected so that its location would not materially influence flow into drains which were installed at practical depths between 2 and 5 feet. The spacing chosen was large enough so that flow rates for drains at different depths would not be significantly affected by this parameter. A drain diameter of 4 inches was used in all analyses since this size of drain is commonly used in field installations.

Some experimental results are shown in figure 10. Drain flow rates were determined for drain depths of 1, 2, · · ·, 8 feet and for different ratios of K_1/K_2. The drain flow rates Q are not given directly but for purposes of generality are expressed in terms of the quantity Q/K_1. Thus the values reported on the abscissa apply to any two-layered soil whose conductivity ratio is given by K_1/K_2, regardless of the absolute values of K_1 and K_2. The flow rate Q is given by simply multiplying Q/K_1 by K_1. If K_1 is 1 foot per day, for example, Q is numerically equal to Q/K_1 and has units of cubic feet of flow daily per foot-length of drain. In the discussion which follows, the quantity Q/K_1 will be referred to as simply "flow rate."

The effect of drain depth on flow rates in an unlayered soil ($K_1 = K_2$) is shown by the curve at the right of figure 10. The solid line represents flow rates as calculated by Kirkham's equation [11] (1949), while the circled points give those evaluated by numerical analysis. The deviation in flow rates as evaluated by these two methods was less than 5%, Kirkham's analytic equation usually yielding the higher values. As shown by Kirkham, flow rates increase with drain

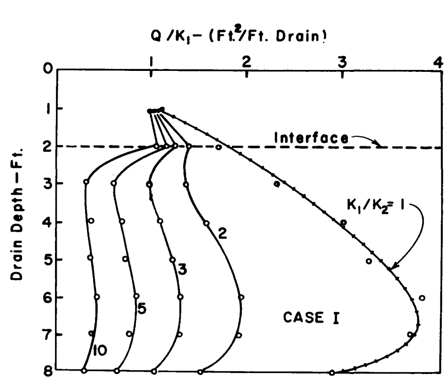

Fig. 10. Drain flow rates as a function of depth of tile line for different ratios K_1/K_2.

depth until the drain approaches the impervious layer. For the 8-foot drain depth, the lower half of the drain is embedded in the impervious layer. However, the sharp decrease in flow rate at the 8-foot drain depth is only partially due to the "half drain" effect since flow rates are also reduced at the 7-foot drain location. The major reason for reduced flow appears to be the restricted flow region adjacent to the drain.

For layered soils, greater flow rates are obtained as the drain location is changed from the center of the top layer to its lower boundary. For ratios of K_1/K_2 equal to or greater than 2, the flow rates are reduced as the drain is first lowered into the bottom layer. There is some increase in flow rates at lower drain locations since flow increases with drain depth. If the ratio K_1/K_2 exceeds 3, the flow rates are less than those obtained when the drain is in the top layer. Apparently greater drain depth does not compensate for the lower conductivity in the bottom layer. As with the unlayered soil, placing the lower half of the drain in the impervious layers reduces the flow rate. However, this reduction is not very marked for ratios greater than 5.

Computer Use in Other Flow Problems

The computer program described here applies only for steady-state flow in saturated soils which are homogeneous with respect to the hydraulic conductivity K. Modified programs can be written, however, for soils which are anisotropic with respect to K. The major difference between programs for isotropic and anisotropic soils involves the approximation formulas. Thus for anisotropic soils, equations [5] through [8] would contain terms which relate the horizontal (K_h) and vertical (K_v) components of the hydraulic conductivity.

Likewise, programs can be written for steady-state flow in unsaturated soil. In such problems the hydraulic conductivity K varies with the moisture content, and the approximation formulas must be altered accordingly. One procedure used by the authors is to assume a unique relationship between water content and the hydrostatic pressure H, and then to relate K to H (for example, Vimoke and Taylor, Eq. [33] and [34]). The approximation formulas would then contain terms whose magnitude is dependent on the pressure H at a particular node and at its four neighboring ones. In such programs, the hydrostatic pressure H at each node is calculated after each iteration by equation [4]. It is then stored in the computer in the same manner as the potential ϕ. Other than the above alteration, the overall procedure would be unchanged from that reported earlier in the text.

The flow problems of general interest in subsurface drainage are time-dependent and involve non-steady-state analysis. Kirkham and Gaskell (1950) have reported a numerical procedure for utilizing a succession of steady-state analyses to determine water-table drawdown in tile and ditch drainage. Their procedure could be easily adapted to computer use, particularly in the more recent models that have larger storage capacity and greater computing speed. The program could be approximately as follows: In the first step, a steady-state analysis is obtained for a water table maintained at the ground surface. This step can be carried out by the program given in the text. In the second step, a small incremental change in moisture content is brought about. This is done first in the top row of nodes (see figure 2) and subsequently at all other nodes where the hydrostatic pressure H indicates unsaturated soil. The change in moisture content is determined by the formula obtained by setting the left side of equation [1] equal to the time-rate change of moisture content $\partial c/\partial t$. In this expression, c is the soil moisture content and t is time. The resulting formula is expressed in finite-difference form, a short time interval is chosen, and Δc is then calculated by assuming that K and ϕ remain unaltered during the chosen time interval.

In the third step, the original moisture content is altered an amount Δc, and the hydrostatic pressure H and potential ϕ are recalculated at each node. The magnitude of H is determined from an experimental or assumed relationship between c and H, while ϕ is calculated by equation [4]. In the fourth step, a second steady-state analysis is obtained by solving equation [1] for the new boundary condition which results from changes in moisture content. In this step, equation [1] applies only to nodes in saturated soil. Steps 1 through 4 are then successively repeated until some arbitrarily chosen time is reached or until the water table has receded to some previously designated depth. At prescribed time intervals, the quantities ϕ, H, and c are printed for each node along with the elapsed drainage time t.

The principal advantage of computer use in such problems is precision and operating speed. Calculations can usually be carried out to 6 or 8 significant figures, while operating speed is exceedingly rapid compared to hand calculation. Availability of the larger and faster computers is steadily increasing from year to year. During the last five years, for example, many institutions have realized a ten- to forty-fold increase in computer operating speed and in storage capacity by replacing existing models with improved ones.

The Researcher's Role in Computer Use

The researcher can and may wish to do his own computer programming. It is not essential that he do so, and in many situations his time will be inefficiently used. In general, the researcher should be able to prepare a flow chart such as illustrated in figure 4. In addition to his familiarity with the problem at hand, the researcher would be greatly assisted in this endeavor by attending a programming seminar which is sponsored by a computer laboratory. Persons adept at programming can usually be employed to convert the flow chart into a suitable language for the machine and to obtain the desired results. Computer laboratories often maintain a directory of personnel who do programming.

SUMMARY

The numerical solution of Laplace's equation by electronic-computer analysis is illustrated for ponded flow in stratified soil. A computer program to solve for the hydraulic-head potential ϕ is presented in detail with the aid of a flow chart. The procedure followed in the program is essentially that reported by Luthin and Gaskell (1950). The primary difference is that a high-speed electronic computer is used instead of a desk calculator. The computer analysis is stopped when the residuals are reduced to a specified value. Finite-difference formulas are given for solving the Laplace equation for the following four cases: square meshes, rectangular meshes, change from square to rectangular meshes, and square meshes containing a curvilinear surface (that is, a circular drain section). The formulas apply to both homogeneous and stratified soils.

Precision is evaluated for a particular mesh size by comparing the potentials at zero residual to those at some finite value of residual. Precision increases nearly linearly with a reduction in the residual. However, computer running time, and thus the cost of these analyses, increases nearly logarithmically with a reduction in residuals. For analyses which require large computer running times, some cost savings can be made by obtaining the potentials for relatively large residuals and then graphically evaluating the potentials at zero residual by extrapolation.

The accuracy of the computer data is evaluated by comparing experimental drain flow rates with those calculated from the exact analytic solutions of Kirkham. Accuracy increases nearly linearly with a reduction in mesh size, and the deviation of computer results from calculated ones ranges between 1 and 2 per cent. Computer running time also increases with a reduction in mesh size, and some compromise is usually made between accuracy and cost.

For a prescribed precision, the use of an overrelaxation constant W materially reduces the computer running time. A technique is presented for deciding on the magnitude of W, and a table of W values is given for a square region. The feasibility of expanding the mesh size in regions where small changes in potential occur is also shown.

Usefulness of the computer program is illustrated by solutions obtained for ponded flow into drains in a stratified soil. The potential use of computers in other subsurface drainage problems is discussed, and a proposal is made for obtaining computer solutions of the falling-water-table case in tile drainage. The role of the researcher in computer usage is also discussed.

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APPENDIX

TABLE A-1

SOME VALUES OF THE CONSTANT C* WHICH ARE USED IN EQUATION [6]†

h_0/r	C	h_0/r	C
1.0000	.00160	2.1818	0.53316
1.0435	.03745	2.4000	0.59412
1.0909	.07299	2.6667	0.66139
1.1429	.10650	3.0000	0.73649
1.2000	.14114	3.4286	0.82158
1.2632	.17660	4.0000	0.91975
1.3333	.21318	4.8000	1.03581
1.4118	.25130	6.0000	1.17784
1.5000	.29124	8.0000	1.36094
1.6000	.33339	9.0000	1.44226
1.7143	.37814	12.0000	1.61897
1.8462	.42598	24.0000	2.06009
2.0000	.47741		

* The values of C reported in this table are equal to one half those reported as C_d by Vimoke and Taylor (1962) in their table 7.

† The C-values are given as a function of the ratio h_0/r, where h_0 is the square-mesh size at the drain and r is the drain radius.

TABLE A-2

SOME OPTIMUM VALUES OF THE OVERRELAXATION CONSTANT W FOR DIFFERENT MATRIX SIZES AS REPORTED BY YOUNG (1954)*

Number of nodes	Overrelaxation constant W
6 by 6 = 36	1.33
10 by 10 = 100	1.53
14 by 14 = 196	1.64
18 by 18 = 324	1.70
22 by 22 = 484	1.75
26 by 26 = 675	1.78
30 by 30 = 900	1.81
34 by 34 = 1,156	1.83
38 by 38 = 1,444	1.85
42 by 42 = 1,764	1.86
∞	2.00

* The values of W are for the case where a square mesh is used and the region of interest is also a square.

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